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Integrable systems from supergravity BPS equations*

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Abstract: Integrable systems of the sine-Gordon/Liouville type, which arise from reducing the BPS equations for solutions invariant under 16 supersymmetries in Type IIB supergravity and M-theory, are shown to be special cases of an infinite family of integrable systems, parametrized by an arbitrary real function f of a real variable. It is shown that, for each function f, this generalized integrable system may be mapped onto a system of linear equations, which in turn may be integrated in terms of the two linearly independent solutions of an ordinary linear second order differential equation which depends only on the function f.

Keywords: Integrable Field Theories, Integrable Equations in Physics.

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Contents

1. Introduction **1**

2. General solution by mapping to a linear system **2**
2.1 A simple example 3

3. Enlarging the integrable system **4**

1. Introduction

A variety of BPS equations for solutions with 16 supersymmetries in Type IIB supergravity in 10 dimensions and in M-theory in 11-dimensions are amenable to general classification [1 − 3], and to exact solution [4 − 7]. Exact solution could be achieved because the BPS equations can be mapped onto integrable two-dimensional field theories of the sine-Gordon, Liouville or Toda type, which in turn can be mapped onto linear problems. In this note, we shall show that the various integrable systems of [5 − 7] naturally fit into an infinite family of integrable systems which are all of the sine-Gordon/Liouville type. Each system is labeled uniquely by a real function f of a single real variable. For each f, we shall exhibit a local map of the integrable system onto a linear system which may be integrated exactly in terms of the two linearly independent solutions of a fixed ordinary linear second order differential equation depending only on the function f.

The integrable systems resulting from the reduced BPS equations in both M-theory and Type IIB supergravity are given in terms of a real field ϑ, and a real harmonic function h, which are both functions on a 2-dimensional Riemann surface Σ with boundary. For the M-theory problem, the field equation for ϑ is given [7] by,

$$\text{M} \qquad 4\partial_{\bar w}\partial_w \vartheta - \partial_{\bar w}\Big(i e^{2i\vartheta} \partial_w \ln h \Big) + \partial_w \Big(i e^{-2i\vartheta} \partial_{\bar w} \ln h \Big) = 0 \tag{1.1}$$

For the Type IIB problem,[1] the field equation for ϑ is given by [5],

$$\text{IIB} \qquad 4\partial_{\bar w}\partial_w \vartheta - \partial_{\bar w}\left(2i \frac{\partial_w h}{\cos(h)} e^{2i\vartheta} \right) + \partial_w \left(2i \frac{\partial_{\bar w} h}{\cos(h)} e^{-2i\vartheta} \right) = 0 \tag{1.2}$$

Both systems may be viewed as special cases of an infinite family of integrable systems of one real field ϑ, labeled by a real function f of one real variable, whose field equation is

$$4\partial_{\bar w}\partial_w \vartheta - \partial_{\bar w}\Big(i e^{2i\vartheta} \partial_w \ln f(h) \Big) + \partial_w \Big(i e^{-2i\vartheta} \partial_{\bar w} \ln f(h) \Big) = 0 \tag{1.3}$$

[1] To exhibit more closely the relation between (1.1) and (1.2), the variables ϑ and μ used in [5] have been redefined by letting $\vartheta \to -2\vartheta + \pi/2$ and $\mu \to h$.

Systems (1.1) and (1.2) respectively correspond to the functions

$$\begin{array}{ll} \text{M} & f(h) = h \\ \text{IIB} & f(h) = \mathrm{tg}\left(\frac{h}{2} + \frac{\pi}{4}\right)^2 \end{array} \tag{1.4}$$

2. General solution by mapping to a linear system

For any function f, the field equation (1.3) is invariant under local conformal transformations of w on Σ. The equation is of the sine-Gordon type in that it involves both types of exponentials $e^{\pm 2i\vartheta}$ (see for example [8, 9], and references therein). In view of the explicit h-dependence, however, the system is also akin to Liouville theory with unbroken time translation invariance (along the direction perpendicular to the coordinate direction h), but broken space translation invariance (parallel to the coordinate direction h) [10 − 12]. In particular, the system shares with Liouville theory the remarkable property that it may be solved exactly (see [10] and references therein), in a manner that will be made clear below.

Integrability of (1.3) for any real function f is guaranteed by the fact that the field equation for ϑ may be recast in the form of a flatness condition,

$$\partial_{\bar{w}}\left(2\partial_w\vartheta - ie^{2i\vartheta}\partial_w \ln f(h)\right) + \partial_w\left(2\partial_{\bar{w}}\vartheta + ie^{-2i\vartheta}\partial_{\bar{w}}\ln f(h)\right) = 0 \tag{2.1}$$

As a result, there must exist a real function ψ, which is locally defined on Σ, such that

$$\begin{aligned} 2\partial_w\vartheta - ie^{2i\vartheta}\partial_w \ln f(h) &= +2i\partial_w\psi \\ 2\partial_{\bar{w}}\vartheta + ie^{-2i\vartheta}\partial_{\bar{w}} \ln f(h) &= -2i\partial_{\bar{w}}\psi \end{aligned} \tag{2.2}$$

Integrability of (2.2), viewed as a system of differential equations for ψ, gives the field equations (1.3) or equivalently (2.1). Integrability in ϑ yields an algebraic relation between ϑ, f and $\partial_w\partial_{\bar{w}}\psi$, which will not be used here.

To integrate the system of first order equations (2.2), the methods of [7] may be generalized to the problem at hand. A simple change of variables allows one to map (2.2) onto a linear problem for any function f. We introduce

$$F \equiv \sqrt{f}\, e^{-\psi - i\vartheta} \tag{2.3}$$

and its complex conjugate $\bar{F}$. In terms of F, the system (2.2) of non-linear first order equations is mapped into a system of linear equations,

$$\partial_w F = \frac{1}{2}(F + \bar{F})\,\partial_w \ln f(h) \tag{2.4}$$

and its complex conjugate. This equation coincides with equation (5.41) of [7], except for the dependence on h, which is now through a general function $f(h)$. Even with this extra f-dependence, (2.4) is manifestly linear, conformal invariant, and may be solved as follows.

Using conformal reparametrization invariance of (2.4) and the fact that h is real and harmonic, we may choose adapted coordinates $w = r + ix$ where $r = h$, and $x = \tilde{h}$ is the

JHEP09(2008)111

harmonic function dual to h, satisfying $\partial_{\bar{w}}(h+i\tilde{h})=0$. Decomposing the complex function $F=F_r+iF_x$ into its real and imaginary parts F_r and F_x, (2.4) decomposes as follows,

$$\begin{aligned}\partial_r F_x - \partial_x F_r &= 0\\ \partial_r F_r + \partial_x F_x &= \frac{f'(r)}{f(r)}\,F_r\end{aligned}\tag{2.5}$$

The first equation in (2.5) may be solved in terms of a single real function $\Phi(r,x)$, and we have $F_r=\partial_r\Phi$ and $F_x=\partial_x\Phi$. In terms of Φ, the second equation becomes,

$$\left(\partial_r^2+\partial_x^2-\frac{f'(r)}{f(r)}\partial_r\right)\Phi(r,x)=0\tag{2.6}$$

Thanks to translation invariance in x, we may solve for the x-dependence of Φ in terms of a Fourier transform in this variable. Using also the reality of the function Φ, the Fourier transform takes the following form,

$$\Phi(r,x)=\int_0^\infty\frac{dk}{2\pi}\Big(\Phi_k(r)e^{-ikx}+\Phi_k^*(r)e^{+ikx}\Big)\tag{2.7}$$

The Fourier modes $\Phi_k(r)$ are complex-vlaued functions of r and k, and must satisfy the following *ordinary* linear second order differential equation,

$$\left(\partial_r^2-k^2-\frac{f'(r)}{f(r)}\partial_r\right)\Phi_k(r)=0\tag{2.8}$$

Denoting the two linearly independent solutions to this equation by $R_k^{(1)}(r)$ and $R_k^{(2)}(r)$, the general solution to (2.8) takes the form,

$$\Phi_k(r)=\phi_1(k)R_k^{(1)}(r)+\phi_2(k)R_k^{(2)}(r)\tag{2.9}$$

in terms of two arbitrary complex functions $\phi_1(k)$ and $\phi_2(k)$. The Wronskian of these two solutions is readily computed, $\partial_r R_k^{(1)}(r)R_k^{(2)}(r)-R_k^{(1)}(r)\partial_r R_k^{(2)}(r)=w_0 f(r)$, were w_0 is found to be independent of r.

2.1 A simple example

A simple, but non-trivial, generalization of (1.1) is obtained by taking $f(h)=h^{2\ell-1}$ for real ℓ. The ordinary differential equation for the Fourier modes then takes the form,

$$\left(\partial_r^2-k^2-\frac{2\ell-1}{r}\partial_r\right)R_k(r)=0\tag{2.10}$$

The linearly independent solutions to this equation may be chosen as follows,

$$\begin{aligned}R_k^{(1)}(r)&=r^\ell I_\ell(kr)\\ R_k^{(2)}(r)&=r^\ell K_\ell(kr)\end{aligned}\tag{2.11}$$

where I_ℓ and K_ℓ are the modified Bessel functions of order ℓ (which need not be an integer). For $\ell=1$, we recover the results of [7]. For ℓ integer, one may also derive this solution via projection of a harmonic function in 3 dimensions, following the methods of [7].

JHEP09(2008)111

3. Enlarging the integrable system

The reduction of the BPS equations in Type IIB supergravity [5], produced the linear equation for $\partial_w F$ of (2.4), as well as a supplementary inhomogeneous linear equation for $\partial_w \bar F$. Translated to the notation of the present paper, this *supplementary equation* takes the form,

$$\partial_w \bar F = \bar F \, \partial_w \ln(1+f) + \kappa \sqrt{f} \tag{3.1}$$

where f is given by equation IIB of (1.4), and $\kappa = \kappa(w)$ is an arbitrary holomorphic 1-form on Σ. The integrability condition between (3.1) and (2.4) is automatically satisfied.

In this section, we shall show that such a supplementary equation exists for all f, and all κ. To derive it, we change variables to a more convenient field G, defined by $F = \sqrt{f} G$. In terms of G, (2.4) takes the form,

$$\partial_w G = \frac{1}{2} \bar G \, \partial_w \ln f(h) \tag{3.2}$$

and we postulate the following Ansatz for the supplementary equation,

$$\partial_w \bar G = g_1 \, G \, \partial_w h + g_2 \, \bar G \, \partial_w h + g_3 \, \kappa \tag{3.3}$$

Here, the functions g_1, g_2, g_3 are complex and depend on h only (but this dependence may involve f), and κ is an arbitrary given holomorphic one-form on Σ. Simultaneous integrability of (3.2) and (3.3) is achieved by requiring that the $\partial_{\bar w}$-derivative of (3.2) and the ∂_w derivative of the complex conjugate of (3.3) be equal, producing the following equations, (we shall denote derivatives with respect to h by a prime, and use the abbreviation $\gamma = f'/f$),

$$\begin{aligned} \gamma^2 &= 4|g_1|^2 + 4\bar g_2' \\ \gamma' &= 2\bar g_1' + 2 g_2 \bar g_1 + \gamma \bar g_2 \\ 0 &= \bar g_1 g_3 \kappa \partial_{\bar w} h + \bar g_3' \bar\kappa \partial_w h \end{aligned} \tag{3.4}$$

We shall show that this system always admits solutions for which the functions g_1, g_2, g_3 are real, and henceforth specialize to this case. The first two equations may be recast as follows,

$$\begin{aligned} 4 g_2' &= (\gamma + 2g_1)(\gamma - 2g_1) \\ (\gamma - 2g_1)' &= g_2 (\gamma + 2g_1) \end{aligned} \tag{3.5}$$

from which we obtain the first integral, $4g_2^2 - (\gamma - 2g_1)^2 = -16\nu^2$, where ν^2 is arbitrary real and independent of h. The solutions to this first integral may be parametrized as follows,

$$\begin{aligned} g_2 &= 2\nu \, \mathrm{sh}(\phi) \\ \gamma - 2g_1 &= 4\nu \, \mathrm{ch}(\phi) \end{aligned} \tag{3.6}$$

while the remaining equation in (3.5) gives $\gamma + 2g_1 = 2\phi'$. The result is a single equation for ϕ in terms of γ, given by $\phi' = \gamma - 2\nu\,\mathrm{ch}(\phi)$. A change of the variable ϕ to $\psi = e^\phi/\sqrt{f}$ leads to an equation of the Riccati type,

$$\psi' = -\frac{\nu}{f}(f^2\psi^2 + 1) \tag{3.7}$$

which may be reduced to a system of two linear equations by the customary change of variables, $\psi = \alpha/\beta$ with α and β real, and we get

$$\begin{aligned} \alpha' &= -\frac{\nu}{f}\beta \\ \beta' &= f\nu\,\alpha \end{aligned} \tag{3.8}$$

The corresponding second order equations solely for α and β are then,

$$\begin{aligned} \alpha'' + \frac{f'}{f}\alpha' + \nu^2\alpha &= 0 \\ \beta'' - \frac{f'}{f}\beta' + \nu^2\beta &= 0 \end{aligned} \tag{3.9}$$

which may be solved in terms of the functions $R_k^{(1,2)}(r)$ introduced in (2.8), upon analytic continuation of the equation under $k \to i\nu$.

It remains to solve the third equation of (3.4). We may assume that $g_3\kappa \neq 0$, since in the contrary case the third equation is manifestly satisfied. There are now two cases,

$$\begin{aligned} &\text{case I} && \kappa\partial_{\bar w}h \pm \bar\kappa\partial_w h \neq 0 \\ &\text{case II} && \kappa\partial_{\bar w}h - \bar\kappa\partial_w h = 0 \end{aligned} \tag{3.10}$$

(A third possibility, for which $\kappa\partial_{\bar w}h + \bar\kappa\partial_w h = 0$, is equivalent to case II.) In case I, we must have $g_1 g_3 = g_3' = 0$, which implies $g_1 = 0$ since $g_3 \neq 0$. This case is easily solved and requires that f be given by an immediate generalization of the expression for the Type IIB case of (1.4), namely $f(h) = c_0\mathrm{tg}(c_1 h + c_2)^2$, with c_0, c_1, c_2 constants, c_0 real, and c_1, c_2 either both real or both imaginary. In case II, we must have $g_1 g_3 + g_3' = 0$, and we may solve for g_3 as follows, $g_3 = g_0\beta/\alpha$.

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JHEP09(2008)111